1. – Given the scalar function of position

$\phi(x,y,z) = x^2y - 3xyz + z^3$

find the value of $\mathrm{grad}(\phi)$ at the point $(3,1,2)$. Also find the directional derivative of (ϕ)at this

point in the direction of the vector $3i - 2j + 6k$

$$\nabla\phi = \frac{\partial\phi(x,y,z)}{\partial r}$$
$$= \frac{\partial r}{\partial}(x^2y - 3xyz + z^3)$$
$$= \frac{\partial x^2y - 3xyz + z^3}{\partial x}\hat{\mathbf{x}} + \frac{\partial x^2y - 3xyz + z^3}{\partial y}\hat{\mathbf{y}} + \frac{\partial x^2y - 3xyz + z^3}{\partial z}\hat{z} = \backslash$$
$$= (2xy - 3yz)\hat{\mathbf{x}} + (x^2 - 3xz)\hat{\mathbf{y}} + (-3xy + 3z^2)\hat{z}$$
$$= (2xy - 3yz, x^2 - 3xz, -3xy + 3z^2)$$

The gradient at the point $(3,1,2)$ is

$$\nabla\phi(3,1,2) = (2 \cdot 3 \cdot 1 - 3 \cdot 1 \cdot 2 - (3)^2 - 3 \cdot 3 \cdot 2, -3 \cdot 3 \cdot 1 + 3 \cdot (2)^2) =$$
$$= (0,-9,3)$$
$$|u| = \sqrt{u^2}$$
$$= \sqrt{u_x^2 \quad + u_y^2 \quad + u_z^2}$$
$$= \sqrt{3^2 + (-2)^2 + 6^2} = \sqrt{9 + 4 + 36} = 7$$

then, the vector is $\hat{u} = (3,-2,6)/7$

So, We can find the direcional deritative

$$\nabla\phi \cdot \hat{u} = (0,-9,3) \cdot \frac{(3,-2,6)}{7}$$
$$= \frac{1}{7}(0.3 + (-9) \cdot (-2) + 3 \cdot 6)$$
$$= \frac{1}{7}(0 + 18 + 18) = \frac{36}{7}$$
$$= \frac{36}{7}$$

2. –If the velocity of a fluid at the point (x,y,z) is given by

$v = (ax + by)i + (cx + dy)j$

find the conditions on the constants a, b, c and d in order that

$$div\ v = 0,\ curl\ v = 0$$

verify that in this case

$$v = \frac{1}{2}\,\mathrm{grad}(ax^2 + 2bxy - ay^2)$$

The velocite of the fluid is $v = (ax + by)i + (cx + dy)j$

Now, we need to find the constans $a,b,c.d$

$divv \equiv \nabla \cdot v = 0$

$curlv \equiv \nabla \times v = 0$

$$\nabla \bullet v = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \bullet (v_{x,} v_y v_z)$$
$$= \frac{\partial v_{x,}}{\partial x} + \frac{\partial v_{y,}}{\partial y} + \frac{\partial v_{z,}}{\partial z} = \frac{\partial (ax+by)}{\partial x} + \frac{\partial (cx+dy)}{\partial y} + \frac{\partial 0}{\partial z}$$
$$= a + b$$
So $a + b = 0 \Rightarrow d = -a$

$$\nabla \bullet v = \begin{pmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_x & v_y & v_z \end{pmatrix}$$
$$= \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z} \right) i - \left(\frac{\partial v_z}{\partial x} - \frac{\partial v_x}{\partial z} \right) j + \left(\frac{\partial v_y}{\partial z} - \frac{\partial v_x}{\partial y} \right) k$$
$$\left(\frac{\partial 0}{\partial y} - \frac{\partial (cx+dy)}{\partial z} \right) i - \left(\frac{\partial 0}{\partial x} - \frac{\partial (ax+by)}{\partial z} \right) j +$$
$$\left(\frac{\partial (cx+dy)}{\partial x} - \frac{\partial (ax+by)}{\partial x} \right) k = 0i + 0j + (c-b)k = (0, 0, c-b)$$
So we have $b - c = 0 \Rightarrow c = b$

$$V = (ax+by)i + bx - ay)j$$
$$v = \mathrm{grad} f \equiv \nabla f$$
$$v_x i + v_y j + v_z z = \frac{\partial f}{ax} i + \frac{\partial f}{ay} j$$
$$\frac{\partial f(x,y)}{\partial x} = v_x$$
$$\frac{\partial f(x,y)}{\partial x} = ax + by$$
$$f(x,y) = \int ax + by dx$$
$$f(x,y) = \tfrac{1}{2} ax^2 + byx + c(y)$$
$$\frac{\partial f(x,y)}{\partial x} = \frac{\partial}{\partial y} \left(\frac{1}{2} ax^2 + byx + c(y) \right)$$
$$= bx + \frac{dC(y)}{dy}$$
$$\frac{dC(y)}{dy} = -ay$$
$$C(y) = \int -anydy$$
$$C(y) = -\frac{1}{2} ay^2 + C$$
$$f(x,y) = \frac{1}{2} ax^2 + byx - \frac{1}{2} ay^2 + C = \frac{1}{2} (ax^2 + 2byx - ay^2 + C)$$